Geometric classification of real ternary octahedral quartics

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PLACEHOLDER LOGO

Objectives

1. The aim is to consider degree four hyper-surfaces invariant under $\mathbf{B_n}$ Coxeter group in $\mathbb{R}^n$.
2. We analyze: octahedral surfaces of degree $\mathbf{4}$.
3. Classify these surfaces geometrically and topologically.
4. study the geometry of the connected components of this family of hyper-surfaces.

Introduction

- The polynomial invariants of finite groups have been studied for more than two centuries and continues to find new applications and generate interesting problems.
- In this work, we are interested in giving a geometric and topologic characterization of real quartic surfaces admitting octahedral symmetry.
- To classify such surfaces we use the same type of group theory approach that was initiated by E. Goursat [4] and extended to the case of icosahedral symmetry by W. Barth [1].

Definitions

- Let $\mathbf{G}$ be a finite group and $\rho : \mathbf{G} \to \mathbf{GL(n, \mathbb{F})}$ a representation of $\mathbf{G}$. Then $\mathbf{G}$ acts on the vector space $\mathbf{V} = \mathbf{F^n}$ through linear transformations, and this may be extended to $\mathbb{F}[\mathbf{V}]$ by the formula $(\mathbf{gf})(\mathbf{v}) := \mathbf{f}(\rho(\mathbf{g}^{-1})\mathbf{v}) \forall \mathbf{v} \in \mathbf{V}$. One of the basic objects of study in invariant theory is the set of $\mathbf{G}$-invariant polynomials $\mathbb{F}[\mathbf{V}]^{\mathbf{G}} := \{\mathbf{f} \in \mathbb{F}[\mathbf{V}] | \mathbf{g}\dot{\mathbf{f}} = \mathbf{f}\ \forall \mathbf{g} \in \mathbf{G}\}$.
- $\mathbb{F}$ is the field of real numbers.
- $\mathbf{G} = \mathbf{B_n}$ the Coxeter group.

Example of a geometric classification for $\mathbf{n = 3}$

Let $\mathbf{n = 3}$. The following equation defines an octahedral quartic.

$$f_{-,-} = \beta(x^2y^2 + y^2z^2 + x^2z^2) - \left(x^2 + y^2 + z^2 + \frac{1}{2}\right)^2 + \frac{1 - k}{4} = 0. \quad (1)$$

1. $\mathbf{0 < \beta \leq 3}$,
   - $\mathbf{k < \frac{3}{3-\beta}}$, $\mathbf{Q_{--}(k) = \mathcal{C}}$, cuboide vanishing in $\{\mathbf{0}\}$.
2. $\beta = \mathbf{3}$
   - $\mathbf{k < 0}$ the surface is an eight branched star
3. $\mathbf{3 < \beta < 4}$
   - $\mathbf{k < \frac{3}{3-\beta}}$, $\mathbf{Q^{--}(k) = (\#_{i=1}^{8}\mathcal{H}_i)\#\mathcal{C}}$, $\#$ The surface is the connected sum of a cube with hyperbolic sheets having the diagonal of the cube as axis.
   - $\mathbf{\frac{3}{3-\beta} = k}$, $\mathbf{Q^{--}(\frac{3}{3-\beta})}$, singular quartic, analgus to the Cayley's cubic: cube with cones at each vertex. this surface has eight conical singular points at the vertices of a cube.
   - $\mathbf{\frac{3}{3-\beta} < k < 0}$, $\mathbf{Q^{--}(k) = (\sqcup_{i=1}^{8}\mathcal{H}_i) \sqcup \mathcal{C}}$.
   - $\mathbf{k = 0}$, $\mathbf{Q^{--}(k) = (\sqcup_{i=1}^{8}\mathcal{H}_i) \sqcup \{0\}}$.
   - $\mathbf{\frac{3}{3-\beta} < 0 < k}$, $\mathbf{Q^{--}(k) = \sqcup_{i=1}^{8}\mathcal{H}_i}$
4. $\beta = \mathbf{4}$,
   - $\mathbf{k < 0}$, $\mathbf{Q_4^{-+}(k) = \sqcup_{i=1}^{6}\mathcal{S}q_i}$, six disjoint hyperbolic sheets with section of square's type $\mathcal{S}\mathbf{q_i}$, and axes being the coordinate axesß†.
   - $\mathbf{k = 0}$, $\mathbf{Q_4^{-+}(0) = \sqcup_{i=1}^{6}\mathcal{S}q_i \sqcup \{0\}}$,
   - $\mathbf{0 < k < 1}$, $\mathbf{Q_4^{-+}(k) = \sqcup_{i=1}^{6}\mathcal{S}q_i \sqcup \mathcal{O}}$ is the disjoint union of six hyperbolic sheets $\mathcal{S}\mathbf{q_i}$ and an octahedron centered at the origin $\{\mathbf{0}\}$.

References

[1] W.Barth *Two projective surfaces with many nodes, admitting the symmetries of the icosahedron* Journal of Algebraic Geometry **5**,(1996), 173-186

[2] R. S. Buringthon. *A classification of quadrics in affine n-space by mean of arithmetic invariant.* Amer. Math. Monthly. **39** ,(1932), 527-532.

[3] J.S.Cassels *An Introduction to the geometry theory of numbers.* Classics in Mathematics Springer (1971)

Results in $\mathbb{R}^3$

- Ased Aliquet Luctus Lectus

Components	**Equation**	**Parameters**	**Compacts**
Components	Equation	Parameters	Compacts
9	$f_{-,-}(x,y,z)$	$3 < \beta, \frac{3}{3-\beta} < k < 0$	1
8	$f_{-,-}(x,y,z)$	$3 < \beta, 0 < k$	0
	$f_{-,+}(x,y,z)$	$0 < \beta < 3, \frac{4}{4-\beta} < k < \frac{3}{3-\beta}$	8
	$f_{-,+}(x,y,z)$	$3 < \beta < 4, \frac{4}{4-\beta} < k$	0
7	$f_{-,+}(x,y,z)$	$\beta = 4, 0 < k < 1$	1
6	$f_{+,+}(x,y,z)$	$\frac{4}{4-\beta} < k < 1$	6
	$f_{-,+}(x,y,z)$	$\beta = 4, k < 0$	0
	$f_{-,+}(x,y,z)$	$4 \leq \beta, k < 0$	0
2	$f_{+,+}(x,y,z)$	$0 < k < \frac{3}{3+\beta}$	2
	$f_{-,+}(x,y,z)$	$0 < \beta < 3, 0 < k < 1$	2
	$f_{-,+}(x,y,z)$	$\beta = 3, 0 < k < 1$	2
	$f_{-,+}(x,y,z)$	$3 < \beta < 4, 0 < k < 1$	1
1	$f_{+,+}(x,y,z)$	$k < 0$	1
	$f_{+,-}(x,y,z)$	$k < 0$	1
	$f_{+,-}(x,y,z)$	$\frac{3}{3+\beta} < k < \frac{4}{4+\beta}$	1
	$f_{-,-}(x,y,z)$	$0 < \beta \leq 3, k < 0$	1
	$f_{-,-}(x,y,z)$	$\beta > 3, 1 < k < \frac{4}{4-\beta}$	0
	$f_{-,+}(x,y,z)$	$0 < \beta \leq 3, k < 0$	1
	$f_{-,+}(x,y,z)$	$0 < \beta \leq 3, 1 < k < \frac{4}{4-\beta}$	1
	$f_{-,+}(x,y,z)$	$\beta = 3, k < 0$	0
	$f_{-,+}(x,y,z)$	$\beta = 3, 1 < k < 4$	0
	$f_{-,+}(x,y,z)$	$3 < \beta < 4, k < 0$	0
	$f_{-,+}(x,y,z)$	$3 < \beta < 4, 1 < k < \frac{4}{4-\beta}$	0
	$f_{-,+}(x,y,z)$	$\beta = 4, 1 < k$	0
	$f_{-,+}(x,y,z)$	$4 \leq \beta, 1 < k$	0

Table 1: Tabular giving the number of connected components in $\mathbb{R}^3$

Figures in $\mathbb{R}^3$

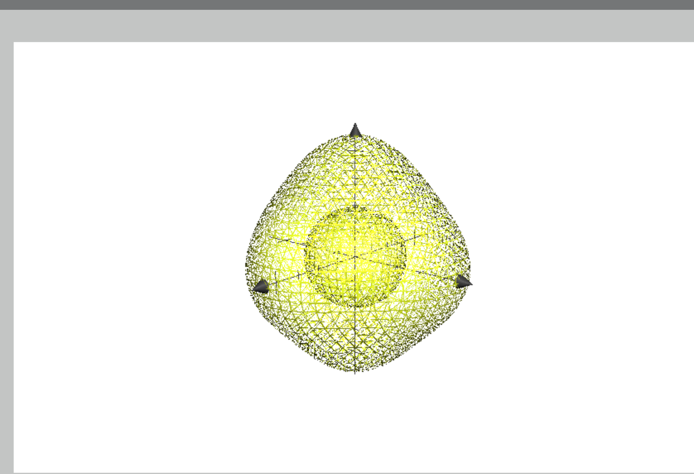

Figure 1: Figure of two nested compact connected components

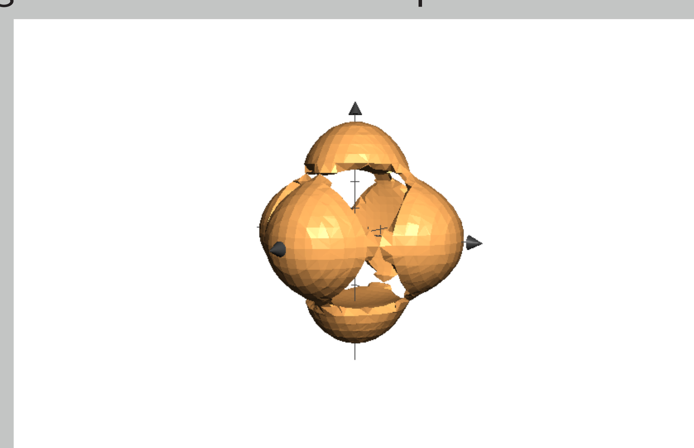

Figure 2: Compact singular surface, 12 conic singularities

Conclusions and results

- Let $\mathbf{f} \in \mathbb{R}[\mathbf{x_1, x_2, \ldots, x_n}]^{\mathbf{B_n}}, \mathbf{deg(f) = 4}$. The octahedral surface defined by $\mathbf{f}$ has at most $\frac{|\mathbf{G}|}{|\mathbf{S_n}|} + \mathbf{1}$ connected components for a given family of parameters.
- The $\mathbf{B_n}$ quartic hyper-surface has at most $\frac{|\mathbf{G}|}{|\mathbf{S_n}|}$ compact connected components.
- If $\mathbf{B_n}$ quartic hyper-surface has two connected components, then they are necessarily nested.
- The maximal bound on the number of connected components of an octahedral quartic surface is $\mathbf{8}$ in $\mathbb{RP}^3$. These compact connected components in $\mathbb{RP}^3$ are homeomorphic to 2-spheres
- The maximal number of isolated singularities of an octahedral quartic is $\mathbf{12}$.

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